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Gravitomagnetism and Gravitational Lens

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Abstract

In [1], we presented a solution to explain dark matter (DM) in the framework of linearized general relativity (GRL) without exotic matter. GRL adds a second component to the Newtonian field, similar to the magnetic field in Electromagnetism (EM), hereinafter called "gravitic field". GRL leads to the Einstein-Maxwell equations which hare equivalent to the Maxwell equations of EM. The aim of [1] was to obtain the expression of the rotation speeds of galaxies far from their center. We then applied this expression to recent observations of the rotation speeds of the Milky Way (MW). This allowed us to obtain a curve in very good agreement with observation and to define the expected value of two gravitic fields (also called gravitomagnetic), the own gravitic field of MW and a uniform external gravitic field embedding MW due to the neighboring clusters of galaxies. In the following article, we will deduce the expression of the Einstein radius of the Einstein rings obtained by gravitational lensing on light rays in this same GRL framework without exotic matter. We will then apply this expression to recent observations on the JWST-ER1 object [3] [4]. Once again, we will obtain results that are very good in agreement with observation since we will find the expected stellar mass within the Einstein ring without the need for any exotic matter. In addition, the gravitic field of JWST-ER1g (the lensing galaxy) will be in agreement with the value of the gravitic field obtained for MW in [1]. This result goes against the criticism that claims that GRL cannot explain the deviation of light and, in particular, the gravitational lensing effect [2], and furthermore, consolidates the solution of [1].

Subject Areas

Cosmology

Keywords

Gravitation, Dark Matter, Galaxies: Fundamental Parameters, Gravitational Lens

1. Introduction

A new solution, obtained from the linearization of General Relativity (GRL), was recently proposed [1] to explain the dark matter (DM) component without exotic matter. GRL adds a second component to the Newtonian field, similar to the magnetic field in Electromagnetism (EM), hereinafter called "gravitic field". GRL leads to the Einstein-Maxwell equations which are equivalent to the Maxwell equations of EM. The solution proposed in [1] explains DM with a gravitic field of the galaxy "$\frac{K_{1\_L}}{r^2}$" greater than that expected by only mass currents and a uniform gravitic field "k_0" due to neighboring clusters of galaxies. Pictorially, the solution of [1] is similar to the situation of particle accelerators in which uniform magnetic fields are essential to maintain particles on high-speed trajectories. This situation in particle accelerators would be very similar to the trajectories of matter at the ends of the galaxies maintained at high speed thanks to this uniform external gravitic field.

One of the criticisms of this GRL solution is that it cannot mimic the deviation of light [2]. In our article, we show on the contrary that the GRL solution with the results calculated in [1] makes it possible to account for Einstein's rings by finding the expected values of stellar mass of lensing galaxies without resorting to exotic matter. We first define the Einstein's radius in this theoretical framework of GRL. It consists in finding the expression of light deviation generated by the gravitic field. And then, we apply our expression to recent observations on the JWST-ER1 object [3] [4].

In [3] and [4], the authors use their measurements of the JWST-ER1 gravitational lens within the framework of the hypothesis of the existence of exotic matter. They then deduce the mass of this exotic matter for the lens galaxy. For [3], this quantity in the Einstein ring is too important to be explained by the "traditional" exotic mass. However, the paper [5] could explain this quantity thanks to the Cold Dark Matter and Self-interacting Dark Matter Interpretations. For [4], this quantity is in agreement with what can be expected from the "traditional" exotic mass hypothesis. In our solution, the measurements of both articles ([3] and [4]) allow us to retrieve the baryonic mass of the lens galaxy without exotic matter. These results will also confirm the gravitic field values obtained to explain the rotation speeds of galaxies in [1] and [6].

2. Expression of the Einstein's radius θ_E and the Baryonic Mass M_{b_L} of the Lensing Galaxy Inner the Einstein Ring

Traditionally, the Einstein's radius θ_E is:

$$\theta_E = \left(\frac{4GM_{Tot\_L}}{c^2} \frac{d_{LS}}{d_S d_L} \right)^{1/2} \tag{1}$$

With c the speed of light, G the gravitational constant, M_{Tot_L} the mass of the lensing galaxy (sum of M_{b_L} the baryonic mass and M_{DM_L} the dark matter), d_L the angular diameter distance to the lens, d_S the angular diameter distance to the source and d_{LS} the distance between the lens and the source.

The field equations of GRL, composed of the two following fields $\vec{g}$ the gravity field and $\vec{k}$ the gravitic field (similar to the magnetic field in electromagnetism), also called the Einstein-Maxwell equations are ([6]-[8]):

$$\begin{aligned}&\vec{g} = -\overrightarrow{grad}\varphi; \vec{k} = \overrightarrow{rot}\vec{H}\\&\overrightarrow{rot}\vec{g} = 0; div\vec{k} = 0;\\&div\,\vec{g} = -4\pi G\rho; \overrightarrow{rot}\vec{k} = -4\pi K^{-1}\overrightarrow{j_p}\end{aligned} \tag{2}$$

With $\varphi = -\frac{GM}{r}$ the Newtonian potential, $\vec{H}$ the vector potential of the gravitic field and $K^{-1} \sim 7.4\times10^{-28}\,m\cdot kg^{-1}$.

These relations are strictly equivalent to those obtained in [7] and [8]. Some coefficients are different only due to the definition of our gravitic field $\vec{k}$. But we remind that the Post-Newtonian formalism can be retrieved with $\vec{k} = \overrightarrow{B_g}/4$, in which $\overrightarrow{B_g}$ is the gravitomagnetic field [9]. The interest of our notation is that the field equations are strictly equivalent to Maxwell idealization with, in particular, a gravitational wave speed of c with the relation $c^2 = (-4\pi G)(-4\pi K^{-1})^{-1}$.

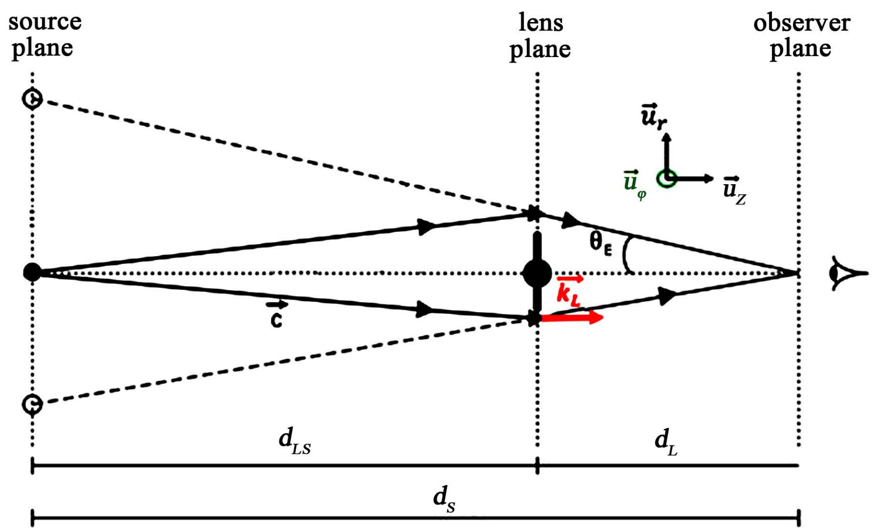

Figure 1. Optical geometry of a gravitational lens where the Einstein's radius θ_E is the observed angle due to lens effect on the apparent source at the observer, d_L is the angular diameter distance to the lens, d_S is the angular diameter distance to the source, and d_{LS} is the angular diameter distance between the lens and the source, $\overrightarrow{k_L}$ the gravitic field in the lens and $\vec{c}$ the light rays of the source.

To simplify our approach, in the cylindrical coordinate system $(\vec{u}_r;\vec{u}_\varphi;\vec{u}_z)$ of a galaxy centered at the center of the galaxy, we assume that we are in the configuration $\vec{k} \perp \vec{u}_r$ and $\vec{v} \perp \vec{u}_r$ (where $\vec{v}$ is the speed of a studied object) and with $\vec{k}$ uniform and $\vec{v}$ with constant speed components. As explained in Appendice, in this configuration and with φ_{b+DM} the Newtonian potential of both baryonic and dark matter and φ_b the Newtonian potential without exotic matter, the GRL modifies the Newtonian potential as:

$$\varphi_{b+DM} = \varphi_b - 4kr\vec{v}.\vec{u}_\varphi . \quad (3)$$

We can notice that this relation is consistent with the relation (20) of [1], which gives the rotational speed at the end of a galaxy.

In our context of light deviation, the speed $\vec{v}$ is now the light celerity $\vec{c}$. Our assumption $\vec{c} \perp \vec{u}_r$ imposes then that the light rays are perpendicular to $\vec{u}_r$. It means that speed of light of the galaxy source passes roughly perpendicularly through the lensing galaxy. If we assume that we see the lensing galaxy in its plane (Fig.1), this assumption is verified. The gravitic field of the lensing galaxy $\overrightarrow{k_L}$ is then roughly along the sight line. And also, contrary to the rotation speed of the galaxy which is along $\vec{u}_\varphi$, in this context the light speed is roughly perpendicular to $\vec{u}_\varphi$ because $\frac{c}{c} \sim \vec{u}_z$. Our relation (3) for the lensing galaxy can then be written around $r \sim r_E$ (radius of Einstein's ring) with $v = c$ the light speed:

$$\varphi_{b\_L+DM\_L}(r_E) = \varphi_{b\_L}(r_E) - (4k_L r_E c)\cos\beta \quad (4)$$

With $\beta = \left(\widehat{\vec{u}_\varphi, \frac{\vec{c}}{c}}\right)$. Given our configuration, it is expected that $\beta \sim 90°$. It will be verified.

By explicitly writing "$\varphi(r) = -\frac{GM}{r}$", we can write:

$$-\frac{G(M_{b\_L} + M_{DM\_L})}{r_E} = -\frac{GM_{b\_L}}{r_E} - (4k_L r_E c)\cos\beta = -\frac{G}{r_E}\left(M_{b\_L} + \frac{(4k_L r_E{}^2 c)\cos\beta}{G}\right) \quad (5)$$

In this configuration, our solution therefore adds to the baryonic mass of the lensing galaxy (M_{b_L}) the equivalent of a "dark matter" to give the following apparent mass (M_{Tot_L}):

$$M_{Tot\_L} = M_{b\_L} + \frac{(4k_L r_E{}^2 c)\cos\beta}{G} \quad (6)$$

And Einstein's radius (1) becomes:

$$\theta_E{}^2 = \frac{4G}{c^2}\frac{d_{LS}}{d_S d_L}\left(M_{b\_L} + \frac{(4k_L r_E{}^2 c)\cos\beta}{G}\right) \quad (7)$$

We can then deduce the mass of the lensing galaxy inner the Einstein ring:

$$M_{b\_L} = \frac{\theta_E{}^2 c^2}{4G}\frac{d_S d_L}{d_{LS}} - \frac{(4k_L r_E{}^2 c)\cos\beta}{G} \quad (8)$$

The 1st member of the right part corresponds to the expression traditionally used [3] [4] to calculate the mass of the lensing galaxy inner the Einstein ring. But in our explanation of the DM, a corrective term (the 2nd member of the right part) appears due to the gravitic field of the lensing galaxy.

Following the relation (9) of [1], the gravitic term k_L in a galaxy is composed of the own gravitic field of the galaxy ($\frac{K_{1\_L}}{r^2}$) and a uniform gravitic field that embeds the galaxy (k_0):

$$k_L(r) = \frac{K_{1\_L}}{r^2} + k_0 \tag{9}$$

Our relation (8) because of (3) was obtained by assuming a uniform field k_L, which is not generally the case because of the gravitic field of the galaxy $\frac{K_{1\_L}}{r^2}$ varying along the galaxy. But the light rays which form the Einstein's ring are all deflected around the distance r_E of the lensing galaxy. In other words, $k_L(r)$ is to be considered only around r_E. We can then consider that the gravitic field of the galaxy changes little around r_E. So, for the deviated rays the hypothesis of a uniform field is verified with good precision and its value is then (around the distance of the Einstein ring radius r_E):

$$k_L(r_E) = \frac{K_{1\_L}}{r_E^2} + k_0 \tag{10}$$

It then gives:

$$M_{b\_L} = \frac{\theta_E^2 c^2}{4G}\frac{d_S d_L}{d_{LS}} - \frac{\left(4\left(\frac{K_{1\_L}}{r_E^2} + k_0\right) r_E^2 c\right)\cos\beta}{G} = \frac{\theta_E^2 c^2}{4G}\frac{d_S d_L}{d_{LS}} - \frac{\left(4K_{1\_L}c\right)\cos\beta}{G} - \frac{\left(4k_0 r_E^2 c\right)\cos\beta}{G} \tag{11}$$

3. Application of Our Relation to the Object JWST-ER1

The study of the object JWST-ER1 has been the subject of at least 2 articles ([3] [4]). These 2 articles agree on the angular diameter of Einstein's ring ($2\theta_E$), the radius r_E that it represents and on the redshift of the lensing galaxy z_l. It allows deducing a value d_L similar for the 2 articles.

For [3]

$$\begin{aligned} &\theta_E \sim \frac{1.54''}{2} \sim 3.733\times10^{-6}\,rad \\ &r_E \sim 6.6kpc \sim 2.037\times10^{20}\,m \\ &z_l \sim 1.94 \end{aligned} \tag{12}$$

We deduce:

$$d_L = \frac{r_E}{\theta_E} \sim \frac{2.037\times10^{20}}{3.733\times10^{-6}} \sim 0.5457\times10^{26}\,m \tag{13}$$

For [4]

$$\begin{aligned} &\theta_E \sim 0.78'' \sim 3.782\times10^{-6}\,rad \\ &r_E \sim 6.6kpc \sim 2.037\times10^{20}\,m \\ &z_l \sim 2 \end{aligned} \tag{14}$$

We deduce:

$$d_L = \frac{r_E}{\theta_E} \sim \frac{2.037\times 10^{20}}{3.782\times 10^{-6}} \sim 0.5386\times 10^{26}\, m \tag{15}$$

On the other hand, the ratio $\frac{d_S}{d_{LS}}$ is quite different in the 2 studies mainly because [3] find the background source at $z_s \sim 2.98$ whereas [4] find $z_s \sim 5.48$.

For [3]

$$\frac{d_S}{d_{LS}} \sim 5.0507 \tag{16}$$

Which gives in their study (*i.e.* without the gravitic field k_L), the total mass in the Einstein ring:

$$M_{Tot\_L} = \frac{\theta_E{}^2 c^2}{4G} d_L \frac{d_S}{d_{LS}} = 6.5\times 10^{11} M_\odot \tag{17}$$

And moreover, they obtain a stellar mass in the Einstein ring:

$$M_{*\_L} = 1.1\times 10^{11} M_\odot \tag{18}$$

For [4]

$$\frac{d_S}{d_{LS}} \sim 2.807 \tag{19}$$

Which gives in their study (*i.e.* without the gravitic field k_L), the total mass in the Einstein ring:

$$M_{Tot\_L} = \frac{\theta_E{}^2 c^2}{4G} d_L \frac{d_S}{d_{LS}} = 3.66\times 10^{11} M_\odot \tag{20}$$

And moreover, they obtain a stellar mass in the Einstein ring:

$$M_{*\_L} = 1.37\times 10^{11} M_\odot \tag{21}$$

Before applying these observation data to our solution, we still need to calculate the value of the gravitic field of the lensing galaxy. To do this, we will rely on the results already obtained in our study [1] on the Milky Way (MW). As said previously, according to this study, we have 2 gravitic field terms, a uniform field k_0 which embeds the universe and the gravitic field of the MW $\frac{K_{1\_MW}}{r^2}$ with the following values:

$$k_0 \sim 10^{-16.65} s^{-1}; \mathcal{K}_{1\_MW} \sim 10^{25.21} s^{-1} m^2 \tag{22}$$

For k_0 it is a uniform field which embeds the Universe on a large scale, from cluster of galaxies to cluster of galaxies [1]. We can therefore expect it to be more or less of the same order of magnitude throughout the Universe [6].

But according to the equations of Einstein-Maxwell [1], even if "K_1" cannot be explained by the mass currents, and in the absence of theory to explain the large value of "K_1" it is not unreasonable to think that this field could be relatively proportional to the mass of the galaxy. Since the observation gives a stellar mass M_{*_L} for the lensing galaxy 3 times greater than that of MW, we can propose a value which will be in order of magnitude:

$$K_{1\_L} \sim 3\times10^{25.21} \sim 10^{25.68} \tag{23}$$

Let's now apply these values at our relation (11).

For the values used in [3], if we take $\beta = 89.9285°$ we obtain for the mass inside the Einstein ring:

$$M_{b\_L} = \frac{\theta_E{}^2 c^2}{4G}\frac{d_S d_L}{d_{LS}} - \frac{(4K_{1_L}c)\cos\beta}{G} - \frac{(4k_0 r_E{}^2 c)\cos\beta}{G} \sim 1.1\times10^{11} M_\odot \tag{24}$$

It corresponds to the mass stellar expected in the Einstein ring. With our solution, no need for exotic matter.

For the values used in [4], if we take $\beta = 89.9685°$ we obtain the mass inside the Einstein ring:

$$M_{b\_L} = \frac{\theta_E{}^2 c^2}{4G}\frac{d_S d_L}{d_{LS}} - \frac{(4K_{1_L}c)\cos\beta}{G} - \frac{(4k_0 r_E{}^2 c)\cos\beta}{G} \sim 1.38\times10^{11} M_\odot \tag{25}$$

It also corresponds to the mass stellar expected in the Einstein ring. Once again, with our solution, no need for exotic matter.

We can note that in the case of [4], the source is further away than [3]. It is consistent with the fact that β is greater and closer to $90°$.

4. Discussion

In our calculation, we can raise a criticism. The main uncertainty on the values concerns the value of K_{1_L} since without a theory to explain the large value of the gravitic fields we deduced K_{1_L} proportionally to the mass of MW from which the expected K_{1_L} value was calculated in [1]. This choice is therefore reasonably open to criticism. It is therefore interesting to study how our solution behaves with respect to different values of K_{1_L}. We can notice that the value of K_{1_MW} that we use must be found in our approximation in a zone with a radius greater than $15kpc$ [1]. In our case, we are at $r_E \sim 6.6kpc$. Our approximation is therefore certainly of poor quality. But we know that in this point idealization the current value of K_{1_L} (which accounts for the speeds far from the center) a priori overestimates the value of the field near the center (since it tends towards infinity in this approximation "$\frac{K_{1\_L}}{r^2}$"). To correct this defect in our modeling, we can therefore expect that using a lower value of K_{1_L} would give a result perhaps closer to reality. If for K_{1_L} we use the value obtained for MW (therefore 3x lower), we obtain for [3], an angle $\beta = 89.7965°$ to obtain their stellar mass and for [4], an angle $\beta = 89.9105°$ to obtain their stellar mass. We thus see that our solution remains stable and coherent despite the uncertainty about the value of K_{1_L}. It is therefore unlikely that this uncertainty could justify a rejection of this GRL solution.

We can also add that our solution in this example appears structurally more stable than the exotic matter hypothesis. Indeed, on the hypothesis of the existence of an exotic mass explaining the DM, the discrepancy $M_{Tot\_L} - M_{*\_L}$ corresponds to the quantity of exotic matter in the Einstein ring. For [3], this quantity in the Einstein ring is too important to be explained by the "traditional" exotic mass. But the paper [5] could explain this difference thanks to the Cold Dark Matter and Self-interacting Dark Matter Interpretations. For [4], this difference gives a quantity in agreement with what can be expected from the "traditional" exotic mass hypothesis. The behavior of exotic matter depending on these two observations can then be very different. Exotic matter solution is then very sensitive in this case to the accuracy of the observations. In our hypothesis of a component of DM explained without exotic matter, but by a gravitic field greater than expected [1] [6], we find these stellar mass values without resorting to an exotic mass and the two results are in agreement with the expected gravitic field. The discrepancy (experimentally due to the uncertainty on the redshift z_s) can be explained by a slight discrepancy in the value β.

5. Conclusions

GRL type solutions can fully explain the dark matter component without exotic matter. But some authors consider that these solutions cannot account for the deviation of light [2]. On the contrary, we have shown concretely in our article that our GRL solution makes it possible to obtain the expected deviation from the only stellar mass for the lensing galaxy, the JWST-ER1 object. In addition, these data confirm the values of the gravitic fields (characteristics of our GRL solution) obtained in several other articles [1] [6] which were necessary to obtain the entire dark matter component to explain the rotation speeds of galaxies. This GRL solution therefore shows very high consistency on these 2 physical problems (flat rotation curves and light deviations share the same value of gravitic field).

Let us remind on this theme of coherence, that these same field values also make it possible to find the MOND theory as an approximation of this GRL solution. Thus, this GRL solution is the only one which allows us to find both the MOND explanation and the exotic matter explanation (see relation (6)). In other words, it is the only solution to date that allows explaining how two alternative solutions, independent of each other, are viable. This is a new strong point in favor of our GRL solution which seems to indicate that it is more fundamental than these 2 alternative solutions (MOND and exotic matter).

Conflicts of Interest

The author declares no conflicts of interest.

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Appendix

In this section, we demonstrate that, with a spatial configuration $\vec{k} \perp \vec{r}$ and $\vec{v} \perp \vec{r}$ and with $\vec{k}$ uniform and $\vec{v}$ of constant speed components, the Newtonian potential φ is modified as the following manner:

$$\varphi \to \varphi - 4kr\vec{v}.\vec{u}_{\varphi} \tag{26}$$

In electromagnetism, when an atom is embedded in a uniform magnetic field $\vec{B}$, as in [10] we can take [10] for the potential vector $\vec{A} = \frac{1}{2}\vec{B} \wedge \vec{r}$. So, let's take for the potential vector:

$$\vec{H} = \frac{1}{2}\vec{k} \wedge \vec{r} \tag{27}$$

We can verify that, with a configuration ($\vec{k} \perp \vec{r}$) and $\vec{k}$ a uniform gravitic field, this definition implies that $\overrightarrow{rot}\ \vec{H} = \vec{k}$ in agreement with Einstein-Maxwell equations. Explicitly, in the cylindrical coordinate system $\left(\vec{u}_r;\vec{u}_{\varphi};\vec{u}_z\right)$ with $\vec{k} \perp \vec{r}$ (whatever φ) and $\vec{k}$ uniform we have the:

$$\vec{k} = \begin{pmatrix} 0 \\ 0 \\ k_z \end{pmatrix} \tag{28}$$

And its potential vector is then:

$$\vec{H} = \begin{pmatrix} H_r \\ H_{\varphi} \\ H_z \end{pmatrix} = \frac{1}{2}\vec{k} \wedge \vec{r} = \frac{1}{2}\begin{pmatrix} 0 \\ 0 \\ k_z \end{pmatrix} \wedge \begin{pmatrix} r \\ 0 \\ 0 \end{pmatrix} = \frac{1}{2}\begin{pmatrix} 0 \\ k_z r \\ 0 \end{pmatrix} \tag{29}$$

It gives

$$\overrightarrow{rot}\vec{H} = \begin{pmatrix} \frac{1}{r}\frac{\partial H_z}{\partial \varphi} - \frac{\partial H_{\varphi}}{\partial z} \\ \frac{\partial H_r}{\partial z} - \frac{\partial H_z}{\partial r} \\ \frac{1}{r}\left(\frac{\partial\left(rH_{\varphi}\right)}{\partial r} - \frac{\partial H_r}{\partial \varphi}\right) \end{pmatrix} = \frac{1}{2}\begin{pmatrix} -\frac{\partial\left(k_0 r\right)}{\partial z} \\ 0 \\ \frac{1}{r}\left(\frac{\partial\left(rk_z r\right)}{\partial r}\right) \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ k_z \end{pmatrix} \tag{30}$$

That shows that we effectively have $\overrightarrow{rot}\vec{H} = \vec{k}$. By this way we verify that the approximation of a uniform gravitic field $\vec{k}$ is compliant with the GRL.

We are now going to demonstrate that in this configuration we have

$$\vec{v} \wedge \left(\overrightarrow{rot}\vec{H}\right) = 2\overrightarrow{grad}\left(\vec{H}.\vec{v}\right) \tag{31}$$

If we assume that, in the previous cylindrical coordinate system, we have a particle of constant speed components and $\vec{v} \perp \vec{r}$, it gives:

$$\vec{v} = \begin{pmatrix} 0 \\ v_{\varphi} \\ v_z \end{pmatrix} \tag{32}$$

We also have $\overrightarrow{grad}\left(\vec{H}.\vec{v}\right)=\frac{1}{2}k_z v_\varphi \vec{u}_r$ because:

$$\vec{H}.\vec{v}=\frac{1}{2}\begin{pmatrix}0\\k_z r\\0\end{pmatrix}\cdot\begin{pmatrix}0\\v_\varphi\\v_z\end{pmatrix}=\frac{1}{2}k_z r v_\varphi \tag{33}$$

and

$$\overrightarrow{grad}\left(\vec{H}.\vec{v}\right)=\begin{pmatrix}\frac{\partial}{\partial r}\\ \frac{1}{r}\frac{\partial}{\partial\varphi}\\ \frac{\partial}{\partial z}\end{pmatrix}\frac{1}{2}k_z r v_\varphi=\begin{pmatrix}\frac{1}{2}k_z v_\varphi\\0\\0\end{pmatrix} \tag{34}$$

Another explicit calculation gives $\vec{v}\wedge\left(\overrightarrow{rot}\vec{H}\right)=k_z v_\varphi \vec{u}_r$:

$$\vec{v}\wedge\left(\overrightarrow{rot}\vec{H}\right)=\vec{v}\wedge\vec{k}=\begin{pmatrix}0\\v_\varphi\\v_z\end{pmatrix}\wedge\begin{pmatrix}0\\0\\k_z\end{pmatrix}=\begin{pmatrix}v_\varphi k_z\\0\\0\end{pmatrix} \tag{35}$$

Finally, in this configuration $\vec{k}\perp\vec{r}$ and $\vec{v}\perp\vec{r}$ and with $\vec{k}$ uniform and $\vec{v}$ of constant speed components, with $\vec{H}=\frac{1}{2}\vec{k}\wedge\vec{r}$ we have:

$$\vec{v}\wedge\left(\overrightarrow{rot}\vec{H}\right)=2\overrightarrow{grad}\left(\vec{H}.\vec{v}\right) \tag{36}$$

By this way, the movement equations of GRL become:

$$\frac{d^2\vec{x}}{dt^2}\approx-\overrightarrow{grad}\varphi+4\vec{v}\wedge\left(\overrightarrow{rot}\vec{H}\right)=-\overrightarrow{grad}\varphi+8\overrightarrow{grad}\left(\vec{H}.\vec{v}\right)\approx-\overrightarrow{grad}\left(\varphi-8\vec{H}.\vec{v}\right) \tag{37}$$

In this configuration, the linearized general relativity modifies the Newtonian potential as:

$$\varphi\rightarrow\varphi-8\vec{H}.\vec{v} \tag{38}$$

With $\vec{k}\perp\vec{r}$ and $\vec{v}\perp\vec{r}$ and with $\vec{k}$ uniform and $\vec{v}$ of constant speed components

$$\varphi\rightarrow\varphi-4kr\vec{v}.\vec{u}_\varphi \tag{39}$$

If we write φ_{b+DM} the Newtonian potential of both baryonic and dark matter and φ_b the Newtonian potential without exotic matter, in our solution we have:

$$\varphi_{b+DM}=\varphi_b-4kr\vec{v}.\vec{u}_\varphi \tag{40}$$